

Spin effects in gravitational radiation back reaction. III. Compact binaries with two spinning components

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(Received 8 May 1998; published 29 October 1998)

The secular evolution of a spinning, massive binary system in eccentric orbit is analyzed, expanding and generalizing our previous treatments of the Lense-Thirring motion and the one-spin limit. The spin-orbit and spin-spin effects up to the 3/2 post-Newtonian order are considered, both in the equations of motion and in the radiative losses. The description of the orbit in terms of the true anomaly parametrization provides a simple averaging technique, based on the residue theorem, over eccentric orbits. The evolution equations of the angle variables characterizing the relative orientation of the spin and orbital angular momenta reveal a speed-up effect due to the eccentricity. The dissipative evolution of the relevant dynamical and angular variables is presented in the form of a closed system of differential equations. [S0556-2821(98)05922-0]

PACS number(s): 04.30.Db, 04.25.Nx

I. INTRODUCTION

The motion of a binary system of two spinning bodies under the influence of gravitational radiation has been treated in several recent works. The aim of these investigations is to obtain a self-consistent description of the evolution, such that templates for the gravitational radiation pattern can be provided for the wave observatories under construction. Spin effects modulate both the amplitude and the frequency of the waves. As of now, the basic properties of the radiative evolution have been understood in a perturbative framework, both in a post-Newtonian expansion [1–6] and by black-hole perturbation techniques [7–9]. The post-Newtonian (PN) expansion proceeds in powers of the parameter $\epsilon \approx v^2/c^2 \approx Gm/c^2 r$. In both approaches, the radiative losses in characteristic quantities, with the inclusion of spin-orbit and spin-spin effects, have been computed. The averages over circular orbits have been obtained in [1] and [2].

The importance of eccentric orbits in various physical scenarios has been emphasized by several authors [10–13]. Quinlan and Shapiro [12] argue that clusters in galactic nuclei in the final stage of collapse will contain a significant number of eccentric binaries. Hills and Bender [13] suggest that many massive ($M \approx 10^6 - 10^7 M_\odot$) compact objects in the galactic centers are gravitationally deflected by others and there is insufficient time left for circularization before plunging. The behavior of eccentric binaries under radiation reaction forces has been studied in [1], [2] and [14], and (neglecting spin effects) in [10]. Averaged radiative losses for eccentric orbits have been obtained for a test particle by Ryan [3]. For finite masses, partial descriptions (by computing the radiation losses of E and L) have been given by Rieth and Schäfer [4]. The averaging procedure for eccentric orbits has proven cumbersome in these works.

In the present series of papers, we investigate the influence of the intrinsic spin on the evolution of a radiating eccentric binary system in a post-Newtonian approach. The presence of spins complicates the description of the orbit considerably; indeed the computations neglecting spins have reached precisions up to the 7/2 PN order [15,16]. In papers

[5] and [6] (to be referred to as **I** and **II**, respectively) we have chiseled two convenient tools for our investigation. The first is the parametrization of the orbit by the generalized *eccentric anomaly* (ξ) and the *true anomaly* (χ) parameters. These parametrizations have already been employed in various special cases. The test-particle limit has been considered in **I** and the one-spin limit in **II**. The second ingredient of our technology is the use of the residue theorem for averaging the gravitational radiation losses by means of these parametrizations. We now further develop and employ both of these tools for the treatment of two spinning masses.

For the purpose of predicting the evolution of a binary with two spins, it is crucial to determine the variation of a complete set of geometrical parameters characterizing not just the orbit but also the orientation of the angular momenta. The amplitude and polarization angle of the gravitational signal is modulated (as has been stressed in [17]) by the changing orientation of the binary in the observer's frame of reference.

In this paper we characterize the motion by computing the evolution of the system parameters to the 3/2 PN order, including spin-orbit and (as an order-of-magnitude analysis will reveal) spin-spin contributions.

The plan of the paper is as follows. In Sec. II, the motion of the binary system is described following [18,19] and [1]. We use the covariant spin supplementary condition (SSC) [2]. Then we introduce the conserved quantities of the non-radiative motion: the energy E and the magnitude of orbital angular momentum L , further the angles subtended by the orbital angular momenta and spins. The evolution of these angles has been discussed for circular orbits in [17], [20] and [21]. Here we give both the instantaneous and averaged evolution equations of these angles, to 3/2 PN order, including the relevant spin-spin terms, for eccentric orbits.

In Sec. III we compute the instantaneous losses of the energy E , magnitude L of the orbital angular momentum and the angle variables from Kidder's universal expressions (in [2]), and by use of the Burke-Thorne [22] potential.

We derive the secular evolution of the system variables due to gravitational radiation reaction (Sec. IV). The contri-

butions from the Burke-Thorne potential have no effect on the averages over a period of revolution. Our results provide a dual description [Eqs. (4.1)–(4.4) in terms of the energy E , orbital angular momentum L and the spin angles vs. Eqs. (4.8), (4.9) in terms of the semimajor axis a , eccentricity e and the spin angles] of the radiating binary system. The results in **I** and **II** are special limiting cases.

II. THE ORBIT OF THE BINARY SYSTEM

The bound state of a two-body system with masses m_1 and m_2 and spins \mathbf{S}_1 and \mathbf{S}_2 , respectively, is described by the Lagrangian

$$\mathcal{L} = \frac{\mu \mathbf{v}^2}{2} + \frac{Gm\mu}{r} + \frac{2G\mu}{c^2 r^3} \mathbf{v} \cdot [\mathbf{r} \times (\mathbf{S} + \boldsymbol{\sigma})] + \frac{\mu}{2c^2 m} \mathbf{v} \cdot (\mathbf{a} \times \boldsymbol{\sigma}). \quad (2.1)$$

Here $r = |\mathbf{r}|$ is the relative distance, \mathbf{v} the relative velocity, μ the reduced mass and m the total mass of the system:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad m = m_1 + m_2, \quad \eta = \frac{m_2}{m_1}. \quad (2.2)$$

The total and weighted spins are defined

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2, \quad \boldsymbol{\sigma} = \eta \mathbf{S}_1 + \eta^{-1} \mathbf{S}_2. \quad (2.3)$$

We have kept only the leading-order spin-orbit coupling, and adopted the spin supplementary condition of [2]. The relative acceleration entering the Lagrangian is

$$\mathbf{a} = -\frac{Gm}{r^3} \mathbf{r} + \frac{G}{c^2 r^3} \left\{ \frac{6}{r^2} \mathbf{r} [(\mathbf{r} \times \mathbf{v}) \cdot (\mathbf{S} + \boldsymbol{\sigma})] - \mathbf{v} \times (4\mathbf{S} + 3\boldsymbol{\sigma}) + 3 \frac{\dot{r}}{r} \mathbf{r} \times (2\mathbf{S} + \boldsymbol{\sigma}) \right\}, \quad (2.4)$$

and the momenta

$$\mathbf{q} = \frac{\partial \mathcal{L}}{\partial \mathbf{a}} = \frac{\mu}{2c^2 m} \boldsymbol{\sigma} \times \mathbf{v}, \quad (2.5)$$

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{v}} - \dot{\mathbf{q}} = \mu \mathbf{v} + \frac{G\mu}{c^2 r^3} \mathbf{r} \times (2\mathbf{S} + \boldsymbol{\sigma}). \quad (2.6)$$

Here an overdot denotes derivative with respect to the time parameter t . The constants of motion are the energy $E = \mathbf{p} \cdot \mathbf{v} + \mathbf{q} \cdot \mathbf{a} - \mathcal{L}$ and the total angular momentum $\mathbf{J} = \mathbf{S} + \mathbf{L}$. The orbital angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} + \mathbf{v} \times \mathbf{q} = \mathbf{L}_N + \mathbf{L}_{SO}, \quad (2.7)$$

with the Newtonian orbital angular momentum and spin-orbit terms

$$\mathbf{L}_N = \mu \mathbf{r} \times \mathbf{v}, \quad (2.8)$$

$$\mathbf{L}_{SO} = \frac{\mu}{c^2 m} \left\{ \frac{Gm}{r^3} [\mathbf{r} \times (\mathbf{r} \times (2\mathbf{S} + \boldsymbol{\sigma}))] - \frac{1}{2} [\mathbf{v} \times (\mathbf{v} \times \boldsymbol{\sigma})] \right\}. \quad (2.9)$$

The spin precession equations have been given in [2]:

$$\begin{aligned} \dot{\mathbf{S}}_1 &= \frac{G}{c^2 r^3} \left(\frac{4+3\eta}{2} \mathbf{L}_N - \mathbf{S}_2 + \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_2) \mathbf{r} \right) \times \mathbf{S}_1, \\ \dot{\mathbf{S}}_2 &= \frac{G}{c^2 r^3} \left(\frac{4+3\eta^{-1}}{2} \mathbf{L}_N - \mathbf{S}_1 + \frac{3}{r^2} (\mathbf{r} \cdot \mathbf{S}_1) \mathbf{r} \right) \times \mathbf{S}_2. \end{aligned} \quad (2.10)$$

The magnitudes S_1 and S_2 of both spins are separately constants.

All spin effects are formally of 1 PN order. A simple estimate $S/L \approx \epsilon^{1/2}$ reveals however that the spin-orbit and spin-spin terms are of respective orders $\epsilon^{3/2}$ and ϵ^2 in general. Exceptionally, in the spin precession Eqs. (2.10) (predicting the evolution of the *directions* $\hat{\mathbf{S}}_i$ of spins), the spin-orbit and spin-spin contributions are of order ϵ and $\epsilon^{3/2}$.

The precession rate of the orbital angular momentum, however, will not include spin-spin terms, as they are of ϵ^2 order. From the conservation of the total angular momentum \mathbf{J} and from (2.10) we obtain

$$\dot{\mathbf{L}} = -\dot{\mathbf{S}} = \frac{G}{2c^2 r^3} (4\mathbf{S} + 3\boldsymbol{\sigma}) \times \mathbf{L}. \quad (2.11)$$

We substituted here, as we did in **II**, \mathbf{L} in place of the Newtonian angular momentum \mathbf{L}_N . It follows from (2.11) that the magnitude L of the orbital momentum and its projection on the vector $4\mathbf{S} + 3\boldsymbol{\sigma}$ are conserved. However, the total spin \mathbf{S} shows a more complicated motion pattern.

The energy and the orbital momentum square are

$$\begin{aligned} E &= \frac{\mu v^2}{2} - \frac{Gm\mu}{r} + \frac{G(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 r^3} \\ &= \frac{\mu}{2} [\dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)] - \frac{Gm\mu}{r} + \frac{G(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 r^3}, \end{aligned} \quad (2.12)$$

$$L^2 = \mu^2 r^4 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) - 4 \frac{G\mu(\mathbf{L} \cdot \mathbf{S})}{c^2 r} + \frac{2E(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 m}. \quad (2.13)$$

Hence we express v^2 and \dot{r}^2 in terms of r , constants of the motion, and in terms of $\mathbf{L} \cdot \boldsymbol{\sigma}$ and $\mathbf{L} \cdot \mathbf{S}$ as follows:

$$v^2 = \frac{2}{\mu} E + \frac{2Gm}{r} - \frac{2G(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 \mu r^3}, \quad (2.14)$$

$$\dot{r}^2 = 2\frac{E}{\mu} + 2\frac{Gm}{r} - \frac{L^2}{\mu^2 r^2} + 2\frac{E(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 m \mu^2 r^2} - \frac{2G}{c^2 \mu r^3} (2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma}). \quad (2.15)$$

Even though the spin projections of \mathbf{L} are not conserved, only the leading-order (conserved) contributions from the quantities $\mathbf{L} \cdot \mathbf{S}_i$ appear in Eqs. (2.14) and (2.15).

To complete the description of the motion, we next compute the evolution of the angle γ subtended by the spins \mathbf{S}_1 and \mathbf{S}_2 , further the angles κ_i subtended by \mathbf{S}_i and \mathbf{L} . In terms of these angles we may express

$$\mathbf{L} \cdot \mathbf{S} = L(S_1 \cos \kappa_1 + S_2 \cos \kappa_2),$$

$$\mathbf{L} \cdot \boldsymbol{\sigma} = L(\eta S_1 \cos \kappa_1 + \eta^{-1} S_2 \cos \kappa_2). \quad (2.16)$$

The variation of the angles κ_i and γ is obtained from Eqs. (2.10) of the spin precession:

$$\begin{aligned} (\cos \kappa_1)' &= \frac{3G(2 + \eta^{-1})}{2c^2 r^3} \frac{S_2}{L} \mathbf{L} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) \\ &\quad + \frac{3G}{c^2 r^5} \frac{S_2}{L} (\mathbf{r} \cdot \hat{\mathbf{S}}_2) \mathbf{L} \cdot (\mathbf{r} \times \hat{\mathbf{S}}_1), \\ (\cos \kappa_2)' &= -\frac{3G(2 + \eta)}{2c^2 r^3} \frac{S_1}{L} \mathbf{L} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) \\ &\quad + \frac{3G}{c^2 r^5} \frac{S_1}{L} (\mathbf{r} \cdot \hat{\mathbf{S}}_1) \mathbf{L} \cdot (\mathbf{r} \times \hat{\mathbf{S}}_2), \\ (\cos \gamma)' &= \frac{3G(\eta - \eta^{-1})}{2c^2 r^3} \mathbf{L} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) \\ &\quad + \frac{3G(\mathbf{r} \cdot \mathbf{S}_2 - \mathbf{r} \cdot \mathbf{S}_1)}{c^2 r^5} \mathbf{r} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2). \end{aligned} \quad (2.17)$$

All terms are of order $\epsilon^{3/2}$, excepting the first term in $(\cos \gamma)'$, which is of order ϵ . Thus in Eqs. (2.17) we need the mixed products only to leading order. First we evaluate

$$\mathbf{L} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) = \mu[(\mathbf{r} \cdot \hat{\mathbf{S}}_1)(\mathbf{v} \cdot \hat{\mathbf{S}}_2) - (\mathbf{r} \cdot \hat{\mathbf{S}}_2)(\mathbf{v} \cdot \hat{\mathbf{S}}_1)],$$

$$\mathbf{L} \cdot (\mathbf{r} \times \hat{\mathbf{S}}_i) = \mu r[r(\mathbf{v} \cdot \hat{\mathbf{S}}_i) - \dot{r}(\mathbf{r} \cdot \hat{\mathbf{S}}_i)]. \quad (2.18)$$

The scalar products may be expressed by use of the angles ψ and ψ_i , subtended by the node line (cf. Fig. 1 in II) with the momentary position vector and the respective projections of the spins in the plane of orbit. These expressions are

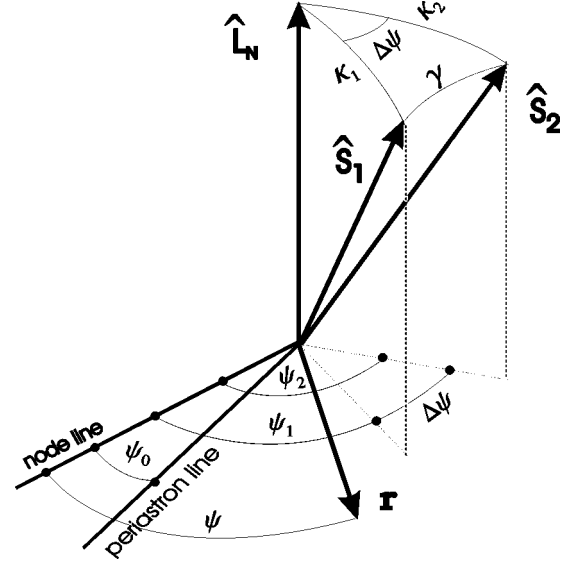


FIG. 1. The angles subtended by $\hat{\mathbf{L}}_N$ and $\hat{\mathbf{S}}_i$, and the difference $\Delta\psi$ of the angles subtended by the node line and the spin projections on the plane of orbit. To leading order, the Newtonian angular momentum $\mathbf{L}_N = \mathbf{L}$. These angles are related by the spherical triangle identity. ψ and ψ_0 are the respective angles of the position \mathbf{r} and the direction of the periastron with the node line.

$$\mathbf{r} \cdot \hat{\mathbf{S}}_i = r \sin \kappa_i \cos(\psi - \psi_i),$$

$$\mathbf{v} \cdot \hat{\mathbf{S}}_i = \dot{r} \sin \kappa_i \cos(\psi - \psi_i)$$

$$- \frac{L}{\mu r} \sin \kappa_i \sin(\psi - \psi_i). \quad (2.19)$$

We have used here the time derivative of the angle ψ i.e., $\dot{\psi} = L/\mu r^2$.

The angles ψ_i do not independently vary. This can be seen by noting that the node line is orthogonal both to the total angular momentum \mathbf{J} and to the Newtonian orbital momentum \mathbf{L}_N . The projection of the total spin on the node line yields

$$S_1 \sin \kappa_1 \cos \psi_1 + S_2 \sin \kappa_2 \cos \psi_2 = 0. \quad (2.20)$$

Hence we may express both angles ψ_i in terms of, say, $\Delta\psi = \psi_2 - \psi_1$. Furthermore, from the spherical cosine identity the angle $\Delta\psi$ is determined by the angles κ_i and γ (Fig. 1):

$$\cos \gamma = \cos \kappa_1 \cos \kappa_2 + \cos \Delta\psi \sin \kappa_1 \sin \kappa_2. \quad (2.21)$$

Thus, in terms of the angles κ_i and $\Delta\psi$ the expressions (2.18), to leading order, take the form

$$\mathbf{L} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) = L \sin \kappa_1 \sin \kappa_2 \sin \Delta\psi,$$

$$\mathbf{L} \cdot (\mathbf{r} \times \hat{\mathbf{S}}_i) = -Lr \sin \kappa_i \sin(\psi - \psi_i). \quad (2.22)$$

The last mixed product in (2.17), again to leading order, works out as

$$\mathbf{r} \cdot (\hat{\mathbf{S}}_1 \times \hat{\mathbf{S}}_2) = r [\cos \kappa_1 \sin \kappa_2 \sin(\psi - \psi_2) - \cos \kappa_2 \sin \kappa_1 \sin(\psi - \psi_1)]. \quad (2.23)$$

It should be stressed that our treatment of the angles is fully coordinate invariant.

We proceed next with the parametrization of the orbit. Following the approach described in Sec. III of **II**, the *eccentric anomaly parameter* ξ is introduced by

$$r = -\frac{Gm\mu}{2E} + \frac{G\mu(2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 L^2} + \left[\frac{A_0}{2E} + \frac{G^2 m \mu^2 (2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 L^2 A_0} - \frac{E(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 m \mu A_0} \right] \cos \xi, \quad (2.24)$$

where A_0 is the length of the Runge-Lenz vector to the zeroth order in the spin:

$$A_0 = \left(G^2 m^2 \mu^2 + \frac{2EL^2}{\mu} \right)^{1/2}. \quad (2.25)$$

As shown in **II**, the orbital period is given by the integral of $dt/d\xi$ from 0 to 2π ,

$$T = 2\pi \frac{Gm\mu^3}{(-2\mu E)^{3/2}}. \quad (2.26)$$

The *true anomaly parameter* χ with properties explained in **I** (and **II**) is introduced similarly,

$$r = \frac{L^2}{\mu(Gm\mu + A_0 \cos \chi)} + \frac{2G(2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 L^2 A_0} \frac{A_0(2G^2 m^2 \mu^3 + EL^2) + Gm\mu(2G^2 m^2 \mu^3 + 3EL^2) \cos \chi}{(Gm\mu + A_0 \cos \chi)^2} - \frac{2E(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2 m \mu^2 A_0} \frac{Gm\mu^2 A_0 + (G^2 m^2 \mu^3 + EL^2) \cos \chi}{(Gm\mu + A_0 \cos \chi)^2}. \quad (2.27)$$

This yields

$$\frac{dt}{d\chi} = \frac{1}{\dot{r}} \frac{dr}{d\chi} = \frac{\mu r^2}{L} \left\{ 1 - \frac{1}{c^2 m L^4} [(2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma}) Gm\mu^2 (3Gm\mu + A_0 \cos \chi) - EL^2 (\mathbf{L} \cdot \boldsymbol{\sigma})] \right\}. \quad (2.28)$$

The leading order expressions of the angle variable ψ and of \dot{r} are also needed in parametrized form:

$$\psi = \psi_0 + \chi, \quad \dot{r} = \frac{A_0}{L} \sin \chi. \quad (2.29)$$

With Eqs. (2.27) and (2.29), all expressions we are interested in are expressed in terms of the true anomaly parameter χ . We integrate these expressions as follows:

$$\int_0^T F(t) dt = \int_0^{2\pi} F(\chi) \frac{dt}{d\chi} d\chi. \quad (2.30)$$

The integration is carried out by computing the residues enclosed in the circle $\zeta = e^{i\chi}$. As a rule, we find that there is only one pole,¹ at $\zeta = 0$.

For a first application, we compute the average rate of change of the angles.

¹We were initially unaware of the conditions under which the true anomaly parameter has this property. A systematic approach has recently been developed [23] to the parametrizations of the perturbed Kepler motion. By its use, we can now prove that the poles of these integrands are in the origin of the complex parameter plane.

$$\begin{aligned}
\langle (\cos \kappa_1) \dot{} \rangle &= \frac{3G(1+\eta^{-1})(-2\mu E)^{3/2}S_2}{2c^2L^3} \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi, \\
\langle (\cos \kappa_2) \dot{} \rangle &= -\frac{3G(1+\eta)(-2\mu E)^{3/2}S_1}{2c^2L^3} \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi, \\
\langle (\cos \gamma) \dot{} \rangle &= \frac{3G(-2\mu E)^{3/2}}{2c^2L^2} \left(\eta - \eta^{-1} + \frac{S_1}{L} \cos \kappa_1 - \frac{S_2}{L} \cos \kappa_2 \right) \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi.
\end{aligned} \tag{2.31}$$

The secular changes do not vanish. This should be contrasted with the behavior of the one-spin system, presented in **II**, where the angular momenta are frozen in a rigidly rotating parallelogram, and the relative angles are constant.

We may express the secular changes of the angles in terms of the semimajor axis $a = (Gm\mu)/(-2E)$ and eccentricity e given by $1 - e^2 = (-2EL^2)/(G^2m^2\mu^3)$. The Keplerian values are sufficient to the accuracy needed here. We then get

$$\begin{aligned}
\langle (\cos \kappa_1) \dot{} \rangle &= \frac{3G(1+\eta^{-1})S_2}{2c^2a^3(1-e^2)^{3/2}} \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi, \\
\langle (\cos \kappa_2) \dot{} \rangle &= -\frac{3G(1+\eta)S_1}{2c^2a^3(1-e^2)^{3/2}} \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi, \\
\langle (\cos \gamma) \dot{} \rangle &= \frac{3G[(\eta - \eta^{-1})\mu(Gma(1-e^2))^{1/2} + S_1 \cos \kappa_1 - S_2 \cos \kappa_2]}{2c^2a^3(1-e^2)^{3/2}} \sin \kappa_1 \sin \kappa_2 \sin \Delta \psi.
\end{aligned} \tag{2.32}$$

In the particular case when the masses and spin magnitudes are equal, $m_1 = m_2$ and $S_1 = S_2$ and with the appropriate change of notation, these equations agree with Apostolatos' [20] Eqs. (4) up to a constant multiplier $1/(1-e^2)^{3/2}$ of the time t . Thus the eccentricity e of the orbit speeds up the evolution of the angular variables κ_i and γ . In the generic case, with arbitrary values of the masses and spin magnitudes, the scaling factor is still present in the denominators of Eqs. (2.31), but no corresponding term in the numerators. We thus find that the eccentricity of the orbit accelerates the evolution of the spin directions.

III. INSTANTANEOUS RADIATIVE LOSSES

We obtain the instantaneous losses of the constants of motion E and L and the angles κ_i and γ using Kidder's [2] results and the Burke-Thorne potential [22]. The variation of the remaining angles follows by the relations (2.20) and (2.21).

First we find the radiative spin losses. To lowest order the radiation-reaction potential is the Burke-Thorne potential [22,24]:

$$V = -\frac{G}{5c^5} I_{\mu\nu}^{(5)} y_\mu y_\nu, \tag{3.1}$$

where $I_{\mu\nu}^{(5)}$ is the fifth time derivative of the system's quadrupole-moment tensor and y_μ are Cartesian coordinates centered on the spinning body.

The radiative spin loss evaluated in **II** holds for each of the spins in its system of principal axes of inertia:

$$\frac{1}{S_i} \frac{d(\mathbf{S}_i)_\mu}{dt} = \frac{2G}{5c^5 \Omega_i} \left(\frac{\Theta_i}{\Theta_i'} - 1 \right) \epsilon_{\mu\nu\rho} I_{\nu\sigma}^{(5)} (\hat{\mathbf{S}}_i)_\rho (\hat{\mathbf{S}}_i)_\sigma. \tag{3.2}$$

Here Θ_i and Θ_i' are the principal moments of inertia and Ω_i is the angular velocity of the i^{th} spinning axisymmetric body. These quantities are related by $S_i = \Theta_i' \Omega_i$.

Two important properties of the radiative spin loss $d\mathbf{S}_i/dt$ emerge from Eq. (3.2):

- (i) they are of 2^{nd} post-Newtonian order and
- (ii) are each perpendicular to the respective spins \mathbf{S}_i .

Next we consider the energy loss dE/dt and total angular momentum loss $d\mathbf{J}/dt$. These have been computed by Kidder using the Blanchet-Damour-Iyer formalism [25]. Keeping the Newtonian and spin-orbit terms in the expressions, we have

$$\frac{dE}{dt} = -\frac{8G^3m^2\mu^2}{15c^5r^4}(12v^2 - 11\dot{r}^2) - \frac{8G^3m\mu}{15c^7r^6} \left[(\mathbf{L}_N \cdot \mathbf{S}) \left(27\dot{r}^2 - 37v^2 - 12\frac{Gm}{r} \right) + (\mathbf{L}_N \cdot \boldsymbol{\sigma}) \left(51\dot{r}^2 - 43v^2 + 4\frac{Gm}{r} \right) \right], \quad (3.3)$$

$$\begin{aligned} \frac{d\mathbf{J}}{dt} = & -\frac{8}{5} \frac{G^2m\mu}{c^5r^3} \mathbf{L}_N \left(-3\dot{r}^2 + 2v^2 + 2\frac{Gm}{r} \right) - \frac{4}{5} \frac{G^2\mu^2}{c^7r^3} \left\{ -\frac{2}{3} \frac{Gm}{r} (\dot{r}^2 - v^2)(\mathbf{S} - \boldsymbol{\sigma}) - \dot{r} \frac{Gm}{3r^2} \mathbf{r} \times [\mathbf{v} \times (7\mathbf{S} + 5\boldsymbol{\sigma})] \right. \\ & + \frac{Gm}{r^3} \mathbf{r} \times \left[(\mathbf{r} \times \mathbf{S}) \left(6\dot{r}^2 - \frac{17}{3}v^2 + 2\frac{Gm}{r} \right) + (\mathbf{r} \times \boldsymbol{\sigma}) \left(9\dot{r}^2 - 8v^2 - \frac{2}{3}\frac{Gm}{r} \right) \right] + \frac{\dot{r}}{r} \mathbf{v} \times \left[(\mathbf{r} \times \mathbf{S}) \left(-30\dot{r}^2 + 24v^2 + \frac{29}{3}\frac{Gm}{r} \right) \right. \\ & + 5(\mathbf{r} \times \boldsymbol{\sigma}) \left(-5\dot{r}^2 + 4v^2 + \frac{5}{3}\frac{Gm}{r} \right) \left. \right] + \mathbf{v} \times \left[(\mathbf{v} \times \mathbf{S}) \left(18\dot{r}^2 - 12v^2 - \frac{23}{3}\frac{Gm}{r} \right) + (\mathbf{v} \times \boldsymbol{\sigma}) \left(18\dot{r}^2 - \frac{35}{3}v^2 - 9\frac{Gm}{r} \right) \right] \\ & \left. + \frac{\mathbf{L}_N}{\mu^2r^2} \left[(\mathbf{L}_N \cdot \mathbf{S}) \left(30\dot{r}^2 - 18v^2 - \frac{92}{3}\frac{Gm}{r} \right) + (\mathbf{L}_N \cdot \boldsymbol{\sigma}) \left(35\dot{r}^2 - 19v^2 - \frac{71}{3}\frac{Gm}{r} \right) \right] \right\}. \quad (3.4) \end{aligned}$$

The loss $dL/dt = \hat{\mathbf{L}} \cdot d\mathbf{J}/dt$ in the magnitude of the orbital angular momentum follows from Eq. (3.4) as the spin-orbit terms in $d\mathbf{J}/dt$ do not receive contributions from the radiative spin losses, cf. property (i). All mixed vector products in Eq. (3.4), when projected to $\hat{\mathbf{L}}$, can be converted to one of $(\mathbf{L} \cdot \mathbf{S})$ and $(\mathbf{L} \cdot \boldsymbol{\sigma})$. The first, Newtonian, term is expressed by use of Eqs. (2.7) and (2.9). Using then Eqs. (2.14), (2.15) and (2.16) we obtain purely radial expressions for the instantaneous losses:

$$\begin{aligned} \frac{dE}{dt} = & -\frac{8G^3m^2}{15c^5r^6}(2\mu Er^2 + 2Gm\mu^2r + 11L^2) + \frac{8G^3mL}{15c^7\mu r^8} [(20\mu Er^2 - 12Gm\mu^2r + 27L^2)(S_1 \cos \kappa_1 + S_2 \cos \kappa_2) \\ & + (6\mu Er^2 - 18Gm\mu^2r + 51L^2)(\eta S_1 \cos \kappa_1 + \eta^{-1} S_2 \cos \kappa_2)], \quad (3.5) \end{aligned}$$

$$\begin{aligned} \frac{dL}{dt} = & \frac{8G^2mL}{5c^5\mu r^5}(2\mu Er^2 - 3L^2) + \frac{8G^2}{15c^7\mu^2r^7} \{ \mu r [12Gm\mu^2Er^2 + 3(G^2m^2\mu^3 + 6EL^2)r - 11Gm\mu L^2](S_1 \cos \kappa_1 \\ & + S_2 \cos \kappa_2) + [2\mu^2E^2r^4 + 12Gm\mu^3Er^3 + 3\mu(G^2m^2\mu^3 + 5EL^2)r^2 - 5Gm\mu^2L^2r + 15L^4](\eta S_1 \cos \kappa_1 \\ & + \eta^{-1} S_2 \cos \kappa_2) \}. \quad (3.6) \end{aligned}$$

We also need the projections of the orbital angular momentum loss in the directions of the spins for evaluating the radiative change in the angle variables. The foregoing considerations yield these projections, after a straightforward but cumbersome computation, in the form

$$\frac{d\mathbf{L}}{dt} \cdot \hat{\mathbf{S}}_i = \frac{dL}{dt} \cos \kappa_i + \frac{2G^2}{15c^7\mu^2r^7} \sum_{j=1}^2 (2p_{ij} + \eta^{3-2j}q_{ij}) S_j \sin \kappa_i \sin \kappa_j \quad (3.7)$$

with

$$\begin{aligned} p_{ij} = & 9\mu L(-2\mu Er^2 + 5L^2)r\dot{r} \sin(\psi_j - \psi_i) + [12Gm\mu^3Er^3 + 3\mu(G^2m^2\mu^3 + 18EL^2)r^2 + 14Gm\mu^2L^2r - 45L^4] \cos(\psi_j - \psi_i) \\ & - 3[4Gm\mu^3Er^3 + \mu(G^2m^2\mu^3 + 6EL^2)r^2 + Gm\mu^2L^2r + 3L^4] \cos(2\psi - \psi_i - \psi_j) + 3\mu L(6\mu Er^2 + 2Gm\mu^2r \\ & - 3L^2)r\dot{r} \sin(2\psi - \psi_i - \psi_j), \quad (3.8) \end{aligned}$$

$$\begin{aligned} q_{ij} = & 15\mu L(-2\mu Er^2 + 5L^2)r\dot{r} \sin(\psi_j - \psi_i) + [4\mu^2E^2r^4 + 24Gm\mu^3Er^3 + 6\mu(G^2m^2\mu^3 + 15EL^2)r^2 + 29Gm\mu^2L^2r - 75L^4] \\ & \times \cos(\psi_j - \psi_i) - [4\mu^2E^2r^4 + 24Gm\mu^3Er^3 + 2\mu(3G^2m^2\mu^3 + 13EL^2)r^2 + 3Gm\mu^2L^2r + 15L^4] \cos(2\psi - \psi_i - \psi_j) \\ & + \mu L(34\mu Er^2 + 12Gm\mu^2r - 15L^2)r\dot{r} \sin(2\psi - \psi_i - \psi_j). \quad (3.9) \end{aligned}$$

These expressions depend on r as well as on the angular variable ψ and on the derivative \dot{r} .

Thus the spin projection of the instantaneous change in $\hat{\mathbf{L}}$ is

$$\frac{d\hat{\mathbf{L}}}{dt} \cdot \hat{\mathbf{S}}_i = \frac{1}{L} \left(\frac{d\mathbf{L}}{dt} \cdot \hat{\mathbf{S}}_i - \frac{dL}{dt} \cos \kappa_i \right) = \frac{2G^2}{15c^7 \mu^2 L r^7} \sum_{j=1}^2 (2p_{ij} + \eta^{3-2j} q_{ij}) S_j \sin \kappa_i \sin \kappa_j. \quad (3.10)$$

Due to property (ii) the instantaneous change in $\hat{\mathbf{S}}_i$ equals the right hand side of Eq. (3.2) and its projection to $\hat{\mathbf{L}}$ yields:

$$\begin{aligned} \hat{\mathbf{L}} \cdot \frac{d\hat{\mathbf{S}}_i}{dt} &= \frac{2G^2 m}{5c^5 \mu^2 r^7 \Omega_i} \left(\frac{\Theta_i}{\Theta'_i} - 1 \right) \sin^2 \kappa_i [4L(-18E\mu r^2 - 20Gm\mu^2 r + 15L^2) \cos(2\psi - 2\psi_i) - \mu r \dot{r} \\ &\quad \times (12E\mu r^2 + 20Gm\mu^2 r + 45L^2) \sin(2\psi - 2\psi_i)]. \end{aligned} \quad (3.11)$$

We are now in position to compute the radiative losses of $\cos \kappa_i$:

$$\frac{d \cos \kappa_i}{dt} = \frac{d}{dt} (\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_i) = \frac{d\hat{\mathbf{L}}}{dt} \cdot \hat{\mathbf{S}}_i + \hat{\mathbf{L}} \cdot \frac{d\hat{\mathbf{S}}_i}{dt}. \quad (3.12)$$

Finally we find from Eq. (3.2) the radiative change in the angle γ subtended by the spin vectors:

$$\begin{aligned} \frac{d \cos \gamma}{dt} &= -\frac{2G^2 m}{5c^5 \mu^2 r^7} \sum_{i \neq j} \frac{1}{\Omega_i} \left(\frac{\Theta_i}{\Theta'_i} - 1 \right) \sin \kappa_i \{ \mu r \dot{r} [(12E\mu r^2 + 20Gm\mu^2 r + 45L^2)(\sin \kappa_i \cos \kappa_j \sin(2\psi - 2\psi_i) \\ &\quad - \cos \kappa_i \sin \kappa_j \sin(2\psi - \psi_i - \psi_j)) + (12E\mu r^2 + 20Gm\mu^2 r - 15L^2) \cos \kappa_i \sin \kappa_j \sin(\psi_j - \psi_i)] + 4L(18E\mu r^2 \\ &\quad + 20Gm\mu^2 r - 15L^2)(\sin \kappa_i \cos \kappa_j \cos(2\psi - 2\psi_i) - \cos \kappa_i \sin \kappa_j \cos(2\psi - \psi_i - \psi_j)) \}. \end{aligned} \quad (3.13)$$

We emphasize that in spite of the fact that the radiative spin losses $d\mathbf{S}_i/dt$ are of 2^{nd} post-Newtonian order, they contribute at $\epsilon^{3/2}$ order to the instantaneous angular losses.

IV. AVERAGED RADIATIVE LOSSES

The instantaneous losses of the constants of motion and the angles subtended by the orbital and spin angular momenta Eqs. (3.5)–(3.13) are in a form suitable for parametrization with the true anomaly parameter χ , using Eqs. (2.27) and (2.29). Then the averaged losses are computed by the residue theorem, passing to the complex variable $\zeta = e^{i\chi}$.

The averaging procedure yields for the constants of motion:

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle &= -\frac{G^2 m (-2E\mu)^{3/2}}{15c^5 L^7} (148E^2 L^4 + 732G^2 m^2 \mu^3 EL + 425G^4 m^4 \mu^6) + \frac{G^2 (-2E\mu)^{3/2}}{10c^7 L^{10}} \{ (520E^3 L^6 + 10740G^2 m^2 \mu^3 E^2 L^4 \\ &\quad + 24990G^4 m^4 \mu^6 EL^2 + 12579G^6 m^6 \mu^9) (S_1 \cos \kappa_1 + S_2 \cos \kappa_2) + (256E^3 L^6 + 6660G^2 m^2 \mu^3 E^2 L^4 \\ &\quad + 16660G^4 m^4 \mu^6 EL^2 + 8673G^6 m^6 \mu^9) (\eta S_1 \cos \kappa_1 + \eta^{-1} S_2 \cos \kappa_2) \} \end{aligned} \quad (4.1)$$

$$\begin{aligned} \left\langle \frac{dL}{dt} \right\rangle &= -\frac{4G^2 m (-2E\mu)^{3/2}}{5c^5 L^4} (14EL^2 + 15G^2 m^2 \mu^3) + \frac{G^2 (-2E\mu)^{3/2}}{15c^7 L^7} \{ (1188E^2 L^4 + 6756G^2 m^2 \mu^3 EL^2 + 5345G^4 m^4 \mu^6) \\ &\quad \times (S_1 \cos \kappa_1 + S_2 \cos \kappa_2) + (772E^2 L^4 + 4476G^2 m^2 \mu^3 EL^2 + 3665G^4 m^4 \mu^6) (\eta S_1 \cos \kappa_1 + \eta^{-1} S_2 \cos \kappa_2) \}. \end{aligned} \quad (4.2)$$

For the angle variables, we first find that Eqs. (3.11) and (3.13) average out:

$$\left\langle \hat{\mathbf{L}} \cdot \frac{d\hat{\mathbf{S}}_i}{dt} \right\rangle = 0, \quad \left\langle \frac{d\gamma}{dt} \right\rangle = 0. \quad (4.3)$$

Thus, when averaging Eq. (3.12), only the first term survives, which leaves us with

$$\begin{aligned} \left\langle \frac{d\kappa_1}{dt} \right\rangle &= \frac{G^2 (-2E\mu)^{3/2}}{30c^7 L^8} \{ (1140E^2 L^4 + 4164G^2 m^2 \mu^3 EL^2 + 2285G^4 m^4 \mu^6) (S_1 \sin \kappa_1 + S_2 \sin \kappa_2 \cos \Delta\psi) \\ &\quad + (884E^2 L^4 + 3264G^2 m^2 \mu^3 EL^2 + 1795G^4 m^4 \mu^6) (\eta S_1 \sin \kappa_1 + \eta^{-1} S_2 \sin \kappa_2 \cos \Delta\psi) \\ &\quad + 6(52E^2 L^4 + 92G^2 m^2 \mu^3 EL^2 + 33G^4 m^4 \mu^6) [S_1 \sin \kappa_1 \cos(2\psi_1 - 2\psi_0) + S_2 \sin \kappa_2 \cos(\psi_1 + \psi_2 - 2\psi_0)] \} \end{aligned}$$

$$+ 2(238E^2L^4 + 431G^2m^2\mu^3EL^2 + 156G^4m^4\mu^6)[\eta S_1 \sin \kappa_1 \cos(2\psi_1 - 2\psi_0) + \eta^{-1}S_2 \sin \kappa_2 \cos(\psi_1 + \psi_2 - 2\psi_0)]\}. \quad (4.4)$$

The corresponding expression $\langle d\kappa_2/dt \rangle$ is obtained by swapping the indices $1 \leftrightarrow 2$ in Eqs. (4.4) and substituting $\eta \leftrightarrow \eta^{-1}$.

For completeness we give the radiative evolution equations also in terms of orbital elements suited for our perturbative treatment. The generalized semimajor axis a and eccentricity e are introduced by $r_{\min}^{max} = a(1 \pm e)$. (The turning points r_{\min}^{max} follow from Eq. (2.24) inserting $\xi = \pi$ and 0 respectively)

$$a = -\frac{Gm\mu}{2E} \left[1 - \frac{2E(2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma})}{c^2mL^2} \right],$$

$$1 - e^2 = -\frac{2EL^2}{G^2m^2\mu^3} \left[1 + \frac{4(2\mathbf{L} \cdot \mathbf{S} + \mathbf{L} \cdot \boldsymbol{\sigma})}{c^2mL^4} (G^2m^2\mu^3 + EL^2) - \frac{2E(\mathbf{L} \cdot \boldsymbol{\sigma})}{c^2mL^2} \right]. \quad (4.5)$$

The inverse relations are

$$E = -\frac{Gm\mu}{2a} \left[1 + \frac{G^{1/2}}{c^2m^{1/2}} \frac{(2 + \eta)S_1 \cos \kappa_1 + (2 + \eta^{-1})S_2 \cos \kappa_2}{a^{3/2}(1 - e^2)^{1/2}} \right],$$

$$L^2 = Gm\mu^2a(1 - e^2) \left\{ 1 - \frac{2G^{1/2}}{c^2m^{1/2}a^{3/2}(1 - e^2)^{3/2}} [S_1 \cos \kappa_1 (3 + e^2 + 2\eta) + S_2 \cos \kappa_2 (3 + e^2 + 2\eta^{-1})] \right\}. \quad (4.6)$$

The first terms on the right-hand sides represent the Keplerian approximation. We emphasize here that the expressions of the angular precessions (2.32) are unchanged if we rewrite them in terms of a and e defined above.

A straightforward computation yields the averaged radiation losses of a , e and κ_i :

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{2G^3m^2\mu(37e^4 + 292e^2 + 96)}{15c^5a^3(1 - e^2)^{7/2}} + \frac{G^{7/2}m^{3/2}\mu}{15c^7a^{9/2}(1 - e^2)^5} \{ (363e^6 + 3510e^4 + 7936e^2 + 2128)(S_1 \cos \kappa_1 + S_2 \cos \kappa_2) + (291e^6 + 4224e^4 + 7924e^2 + 1680)(\eta S_1 \cos \kappa_1 + \eta^{-1}S_2 \cos \kappa_2) \}, \quad (4.7)$$

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{G^3m^2\mu e(121e^2 + 304)}{15c^5a^4(1 - e^2)^{5/2}} + \frac{G^{7/2}m^{3/2}\mu e}{30c^7a^{11/2}(1 - e^2)^4} \{ (1313e^4 + 5592e^2 + 7032)(S_1 \cos \kappa_1 + S_2 \cos \kappa_2) + (1097e^4 + 6822e^2 + 6200)(\eta S_1 \cos \kappa_1 + \eta^{-1}S_2 \cos \kappa_2) \}, \quad (4.8)$$

$$\left\langle \frac{d\kappa_1}{dt} \right\rangle = \frac{G^{7/2}m^{3/2}\mu}{30c^7a^{11/2}(1 - e^2)^4} \{ (285e^4 + 1512e^2 + 488)(S_1 \sin \kappa_1 + S_2 \sin \kappa_2 \cos \Delta\psi) + (221e^4 + 1190e^2 + 384)(\eta S_1 \sin \kappa_1 + \eta^{-1}S_2 \sin \kappa_2 \cos \Delta\psi) + (156e^4 + 240e^2)[S_1 \sin \kappa_1 \cos(2\psi_1 - 2\psi_0) + S_2 \sin \kappa_2 \cos(\psi_1 + \psi_2 - 2\psi_0)] + (119e^4 + 193e^2)[\eta S_1 \sin \kappa_1 \cos(2\psi_1 - 2\psi_0) + \eta^{-1}S_2 \sin \kappa_2 \cos(\psi_1 + \psi_2 - 2\psi_0)] \}. \quad (4.9)$$

The substitutions following Eq. (4.4) should be carried out once more to get the secular variation of κ_2 .

The averaged losses (4.1), (4.2), (4.3) and (4.4) (and the similar expression for $\langle d\kappa_2/dt \rangle$), together with the algebraic relations (2.20) and (2.21), provide a complete description of the radiative evolution of the binary system, to $\epsilon^{3/2}$ order in terms of radiative losses. On the other hand the radiative changes of the angles κ_i are of $\epsilon^{5/2}$ relative order compared to their secular changes given in Eq. (2.32). An alternative set of evolution equations, in terms of the orbital elements a ,

e and κ_i , is provided by Eqs. (2.32), (4.7), (4.8) and (4.9). It is remarkable that all the contributions from the radiative losses of the spins, present in the instantaneous losses average to zero.

V. CONCLUDING REMARKS

Our description of the smoothed evolution of a spinning binary system opens up the possibility to study the modulation of the gravitational wave forms induced by the eccen-

tricity. Investigations of the smoothed evolution of circular orbits have already been presented in several papers ([17], [20] and [21]). The detailed analysis of the angular evolution equations is a subtle issue. An immediate effect that follows from Eqs. (2.31) of Sec. II is the acceleration of the evolution of spin orientations with increasing eccentricity.

The radiative losses of both the dynamical quantities E and L and of the angular variables κ_i and γ subtended by the angular momenta were given here up to $\epsilon^{3/2}$ order compared to the leading order losses. Among them the angular losses of the angles κ_i are of $\epsilon^{5/2}$ order beyond the secular spin-orbit terms given in Eq. (2.32). Our previous results in **I** and **II** are particular cases of the present radiative loss equations. The one-spin limit arises by $S_2 \rightarrow 0$ and $\psi_1 = \pi/2$ [the latter relation stems from Eq. (2.20)]. For the Lense-Thirring case the additional limit $\eta \rightarrow 0$ has to be taken. We would also

like to point out the agreement of the energy and orbital momentum losses with computations in a different, noncovariant SSC. There the energy E and orbital momentum L were derived from a different action, but their radiative losses, given by Rieth and Schäfer [4] coincide with those in our Eqs. (4.1) and (4.2). However, in our approach, the previously unknown radiative evolution of the angles κ_i and γ , characterizing the geometry of the binary system, could readily be obtained.

ACKNOWLEDGMENTS

This work has been supported by OTKA No. T017176 and D23744 grants. The algebraic package REDUCE was used for checking our computations.

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